

GAUSSIAN QUADRATURE OF $\int_0^1 f(x) \log^m(x) dx$ AND $\int_{-1}^1 f(x) \cos(\pi x/2) dx$

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ABSTRACT. We tabulate the abscissae and associated weights for numerical integration of integrals with either the singular weight function $(-\log x)^m$ for exponents $m = 1, 2$ or 3 , or the symmetric weight function $\cos(\pi x/2)$. Standard brute force arithmetics generates explicit pairs of these values for up to 128 nodes.

1. METHODOLOGY

The paper provides abscissae x_i and weights w_i for Gaussian integration with a power of a logarithm in the integral kernel on one hand,

$$(1) \quad \int_0^1 f(x) (-\log x)^m dx \approx \sum_{i=1}^N w_i f(x_i).$$

or with a cosine in the integral kernel on the other,

$$(2) \quad \int_{-1}^1 f(x) \cos(\pi x/2) dx \approx \sum_{i=1}^N w_i f(x_i).$$

The w_i and x_i are computed with the standard theory from roots of a system of orthogonal polynomials p_n with norm [6, 12, 14]

$$(3) \quad \langle f, g \rangle \equiv \int_0^1 f(x) g(x) (-\log x)^m dx,$$

and

$$(4) \quad \langle f, g \rangle \equiv \int_{-1}^1 f(x) g(x) \cos \frac{\pi x}{2} dx,$$

respectively. A set of orthogonal (monic) polynomials $p_n(x)$ is bootstrapped from

$$(5) \quad p_{-1}(x) = 0; \quad p_0(x) = 1; \quad p_{n+1}(x) = (x - a_n)p_n(x) - b_n p_{n-1}(x).$$

[Dependence of polynomials and coefficients a and b on the parameter m in the case (1) is not written down explicitly here.] Multiplication of the recurrence with p_n or p_{n-1} and using the requirement of orthogonality proposes to calculate the coefficients and polynomials recursively with

$$(6) \quad a_n = \frac{\langle x p_n, p_n \rangle}{\langle p_n, p_n \rangle};$$

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$$(7) \quad b_0 = 0; \quad b_n = \frac{\langle xp_n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} \quad (n > 0).$$

Remark 1. In cases like (2) where the weight in the integral is an even function and the integral limits are symmetric, all a_n are zero.

The standard further steps are

- normalization of the polynomials such that their norm is unity,

$$(8) \quad p_n^*(x) \equiv \frac{p_n(x)}{\sqrt{\langle p_n, p_n \rangle}},$$

- computation of all zeros x_i of $p_N(x)$ at some degree N .
- computation of the weights w_i by

$$(9) \quad w_i = -\frac{[x^{N+1}]p_{N+1}^*}{[x^N]p_N^*} \frac{1}{p_{N+1}^*(x_i)p_N^{*'}(x_i)},$$

where $[x^{N+1}]p_{N+1}^*$ and $[x^N]p_N^*$ are the leading coefficients of the two polynomials after normalization, and where the prime at p' denotes the derivative with respect to x .

We obviously add no new aspect to the established theory. The benefit is to those readers who need explicit abscissae-weight pairs and have no access to a multi-precision numeric library.

2. LOGARITHMIC KERNEL

The first part of the results extends tables that have been published in the literature for exponent $m = 1$, namely by Anderson for N up to 10 [2], by Danloy for $N = 10$ and $N = 20$ [5], and by King for $N = 20$ and $N = 30$ [9].

Remark 2. The variable substitution $x = e^{-y}$ changes the format to

$$(10) \quad \int_0^1 f(x)(-\log x)^m dx = \int_0^\infty f(e^{-y})y^m e^{-y} dy$$

which is alternatively evaluated with Gauss-Laguerre quadratures [1, (25.4.38)][3, 13].

Integrals of the form (3) are calculated for the polynomials that appear in the recurrence (5) term-by-term with the aid of the moments μ [7, 2.722],

$$(11) \quad \mu_{n,m} \equiv \int_0^1 x^n (-\log x)^m dx = \frac{m!}{(n+1)^{m+1}}.$$

The first polynomials $p_{n,m}(x)$ look as follows:

$$\begin{aligned}
(12) \quad p_{1,1} &= x - 1/4; \\
(13) \quad p_{2,1} &= x^2 - 5/7 x + \frac{17}{252}; \\
(14) \quad p_{3,1} &= x^3 - \frac{3105}{2588} x^2 + \frac{5751}{16175} x - \frac{4679}{258800}; \\
(15) \quad p_{1,2} &= x - 1/8; \\
(16) \quad p_{2,2} &= x^2 - \frac{19}{37} x + \frac{217}{7992}; \\
(17) \quad p_{3,2} &= x^3 - \frac{1632663}{1695176} x^2 + \frac{5619807}{26487125} x - \frac{1568083}{242168000}; \\
(18) \quad p_{1,3} &= x - 1/16; \\
(19) \quad p_{2,3} &= x^2 - \frac{13}{35} x + \frac{493}{45360}; \\
(20) \quad p_{3,3} &= x^3 - \frac{129197997}{166534960} x^2 + \frac{4147011999}{32526359375} x - \frac{19126701359}{8326748000000}.
\end{aligned}$$

$p_{2,1}$ in particular has been written down earlier [11]. Two generic values are

$$\begin{aligned}
(21) \quad p_{1,m} &= x - 2^{-1-m}; \\
(22) \quad p_{2,m} &= x^2 + \frac{-2^{m+1} + 3^{m+1}}{3^{m+1} - 4^{m+1}} x + \frac{-3^{m+1} 2^{-1-m} + 4^{m+1} 3^{-1-m}}{3^{m+1} - 4^{m+1}}.
\end{aligned}$$

The results are summarized in the ASCII files `log_N_m` in the ancillary directory, where N covers the range 3 to 128 and m covers powers from 1 to 3. Each line contains a pair (x_i, w_i) . For improved readability, a blank line is inserted after each block of 5 nodes. The numbers have been stabilized to the 30 digits shown by cranking up the internal representation of numbers in a Maple program to 270 digits.

Remark 3. *Related approximative cubatures where polynomials are not only multiplied by also added to the logarithm in the kernel have also been discussed [8, 4, 10].*

3. COSINE KERNEL

The tools to assemble (2) start from repeated partial integration of [7, 3.761]

$$(23) \quad \int_0^{\pi/2} x^m \cos x dx = \sum_{k=0}^{\lfloor m/2 \rfloor} (-)^k \frac{m!}{(m-2k)!} \left(\frac{\pi}{2}\right)^{m-2k} + (-)^{\lfloor m/2 \rfloor} m! (2 \lfloor \frac{m}{2} \rfloor - m),$$

for non-negative integer m . The even moments are therefore

$$\begin{aligned}
(24) \quad \mu_{2m} &\equiv \int_{-1}^1 x^{2m} \cos(x\pi/2) dx = 2(2m)! \sum_{k=0}^m (-)^k \frac{1}{(2m-2k)!} (2/\pi)^{2k+1} \\
&= \frac{4}{\pi} {}_3F_0 \left(\begin{matrix} -m + \frac{1}{2}, -m, 1 \\ - \end{matrix} \mid -\frac{16}{\pi^2} \right).
\end{aligned}$$

The odd moments are zero because the cosine is an even function. The monic orthogonal polynomials start

$$(25) \quad p_0 = 1; \quad p_1 = x;$$

$$(26) \quad p_2 = x^2 - 1 + \frac{8}{\pi^2};$$

$$(27) \quad p_3 = x^3 - \frac{\pi^4 - 48\pi^2 + 384}{(\pi^2 - 8)\pi^2}x;$$

$$(28) \quad p_4 = x^4 - 2\frac{\pi^4 - 78\pi^2 + 672}{\pi^2(\pi^2 - 10)}x^2 + \frac{\pi^6 - 114\pi^4 + 1728\pi^2 - 6912}{\pi^4(\pi^2 - 10)},$$

and have parities $p_{-n}(x) = (-1)^n p_n(x)$.

The results are summarized in the ASCII files `cosine_N` in the ancillary directory, where N covers the range 3 to 128. The numbers have been stabilized to the 30 digits shown by an internal representation of numbers in a Maple program with 650 digits.

Only the values with positive x_i or $x_i = 0$ are tabulated; the duplicates of the nodes at the negative abscissae (with the same weights) are not added explicitly.

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